<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Exploration of Graham’s Conjecture in Graph Pebbling</th>
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<tbody>
<tr>
<td><strong>Name of PI</strong></td>
<td>Charles Cusack and Airat Bekmetjev</td>
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<td><strong>Department of PI</strong></td>
<td>Computer Science and Mathematics</td>
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<tr>
<td><strong>Name(s) of collaborators</strong></td>
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<tr>
<td><strong>Undergraduates Associated with Project:</strong> (Give names if known or simply numbers if students are not yet identified)</td>
<td>2 students during summer 2015</td>
</tr>
<tr>
<td><strong>Projected start date</strong></td>
<td>May 2015</td>
</tr>
<tr>
<td><strong>Projected end date</strong></td>
<td>July 2015</td>
</tr>
<tr>
<td><strong>Total Budget Requested</strong></td>
<td>$14,961</td>
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ABSTRACT (keep the abstract under ½ of a page)

Certain network optimization problems can be modeled using graph pebbling. Graph pebbling problems have sparked the interest of numerous researchers in mathematics and computer science, with an increasing number of published papers in the area each year. Graham’s conjecture about the pebbling number of the Cartesian product of two graphs has been of particular interest and has been one of the driving forces in the further development of the field. The goal of this project is to analyze problems related to Graham’s conjecture. This will involve creating more efficient algorithms to determine pebbling solvability of certain classes of graphs as well as studying several specific graphs (e.g. the Lemke graph) in considerable detail in order to better understand their relationship to the conjecture. The project will involve two faculty members and two Hope College students during the summer of 2015.
1 Project Description

1.1 Significance of Work

Pebbling is a process defined on a connected graph. A graph is a set of vertices and edges that connect some of the vertices. A connected graph is a graph where there is a path between any two vertices (see Figure 1). Graphs can be used to model transportation and communication networks. Pebbles, which can represent resources or information packets, are placed on the vertices of a connected graph and are moved along the edges. A move consists of taking two pebbles from one vertex, placing one pebble on an adjacent vertex, and discarding the other as a cost of transfer. The objective of pebbling is to put at least one pebble on a designated vertex, called the root. A configuration (that is, a placement of pebbles on the vertices) is called r-solvable if it is possible to move a pebble to the root r. A configuration is called solvable if it is r-solvable for every vertex r in the graph. For example, the configuration in Figure 1 is solvable. The pebbling number of a graph G, denoted \( \pi(G) \), is the minimum number \( t \) such that every configuration of \( t \) pebbles is solvable.

The problem of determining the solvability of a graph is in a class of problems known as NP-complete [1-2]. NP-Complete is a class of mathematical problems for which an efficient algorithm (that is, one that runs in polynomial time) has not been established, nor has anyone proven that such an algorithm does not exist. Further, an efficient algorithm to solve any one NP-complete problem would imply an efficient algorithm for all of them. Put simply, this implies that there is likely no efficient algorithms to determine pebbling solvability of graphs in general. However, it is possible to establish efficient algorithms that work well for specific classes of graphs based on unique structural properties of these graphs (e.g. planar graphs) [3].

Graph pebbling is related to various optimization problems on networks [4-7]. One of the most important problems in graph pebbling is a conjecture about the properties of the Cartesian product of graphs known as Graham’s conjecture. The Cartesian product of two graphs \( G \) and \( H \), denoted as \( G \square H \), is the graph that replaces each vertex of graph \( G \) by a copy of the entire graph \( H \). The vertices from each of these copies of \( H \) are connected to the corresponding vertices in the other copies of \( H \) according to the edges from \( G \). Graham’s conjecture states that for two graphs \( G \) and \( H \), \( \pi(G \square H) \leq \pi(G) \times \pi(H) \). That is, the pebbling number of the Cartesian product of two graphs is no larger than the product of the pebbling numbers of the two graphs. This conjecture was confirmed on several families of graphs such as cubes, cycles, trees, and graphs with special properties such as the 2-pebbling property [8-11]. The research related to Graham’s conjecture
inspired many directions of study in graph pebbling. It motivated the development of the field and resulted in many generalizations of the pebbling concepts and predicted many theoretical results.

1.2 Objectives
The main objective of the project is to analyze problems related to Graham’s conjecture from algorithmic and computational directions. We will improve the state of the art in terms of pebbling solvability algorithms and explore various related problems that were used in the analysis of the Graham’s conjecture (such as the 2-pebbling property, Lemke graph etc.).

The other main objective is to provide undergraduate students with interdisciplinary research experience in advanced areas of mathematics and computer science. Pebbling on graphs is a rapidly developing area of research that uses cutting edge methods from computer science as well as areas of mathematics such as combinatorics, graph theory, probability theory and discrete mathematics. The students will have an opportunity to be involved in actual scientific research in an actively developing area that will prepare them for future research careers in these fields.

1.3 Methods
We will build on our recent success in solving other graph pebbling problems [3, 12-13] and will leverage the Algoraph framework that we developed over the past several years to assist in solving a variety of graph-related problems [14-18].

A promising approach to better understand Graham’s conjecture that was suggested in recent years is to analyze products of graphs that do not exhibit the 2-pebbling property. The minimal such graph is called the Lemke graph. It was proposed that the analysis of the pebbling solvability of the Cartesian product of two Lemke graphs, denoted $L^2$ (that is, $L^2 = L \Box L$) will be crucial for the validity of Graham’s conjecture.

As is the case with many classes of graphs, the analysis of $L^2$ is computationally very complex and there are currently no efficient algorithms that can analyze the problem in a reasonable (i.e. polynomial) amount of time. Therefore we will use techniques such as linear programming, branch-and-bound, and decrease-and-conquer (e.g. vertex elimination) to improve existing Algoraph algorithms and create new ones that are sufficient to examine the pebbling solvability of $L^2$ and certain other classes of graphs. We will use the structural properties of these graphs to significantly reduce the number of cases the algorithms will need to consider. The algorithms will consider numerous target pebbling configurations and analyze their solvability and will be applicable to a wider range of graphs than just the specific ones we will be focusing on for this project.

At the same time we will work on the general proof of Graham’s conjecture as well as the identification of new families of graphs for which we will prove that the conjecture is true. We will begin by studying all of the techniques used in the previous partial results and cataloging all of the cases for which Graham’s conjecture is already known to hold. We will then try to generalize or extend the arguments to either prove the conjecture for all cases or prove it for
additional general classes of graphs (e.g. based on their diameter, maximum vertex degree, regularity, or other such properties). We will also consider the concept of fractional pebbling that has been proven to be useful in determining solvability of pebbling configurations.

1.4 Expected Outcomes
We will improve existing graph pebbling algorithms and develop new algorithms. We will have a full understanding of the current state of Graham’s conjecture. We expect that we will be able to confirm or rule out Graham’s conjecture for certain graphs (such as $L^2$), as well as prove the result for new families of graphs.

The students will have the experience of being involved in the entire scientific process beginning from stating research hypotheses, carrying out the analysis and producing the theoretical statements, and then implementing the new algorithms to verify their findings. The students will be the leading authors on a research paper with the results of their work. We will disseminate the result this project and the developed methods and tools to the scientific community and encourage other researchers to use them. This will be done through a journal article and/or presentation of our results at conferences such as MathFest or the annual SIGCSE (Special Interest Group on Computer Science Education) conference.

The successful completion of this project will stimulate interdisciplinary curriculum development. The developed algorithms and theoretical findings will be included in courses such as in Discrete Mathematics and Advanced Algorithms offered by the departments of Mathematics and Computer Science at Hope College.

1.5 Potential Difficulties
A potential challenge is that it is hard to predict how long it will take to run an algorithms on a given graph. If the time is much longer than we had anticipated, we will mitigate this by using multiple PCs available to the computer science department and potentially the Curie cluster (a parallel computing cluster housed at the Science Center).

The second potential difficulty is that our attempt to prove Graham’s conjecture will establish that it is true for some specific cases rather than produce a general proof. If this is the case, it will definitely enhance the body of knowledge in this area of research and will provide helpful insights for other researchers in the field who are working on this problem. Adding new general classes of graph for which Graham’s conjecture is true can eventually become an indirect proof of the statement if all possible cases can be considered. In particular, proving that it holds for $L^2$ would be a very strong evidence that the conjecture is true in general.

1.6 Connection to other HHMI Programs
Dr. Cusack has worked with two to six students each summer from 2006-2013 on several different projects, including several related to graph pebbling. These students have been involved in writing two journal articles [3, 12], three peer-reviewed conference papers [14-16], and four poster and software presentations at professional conferences [17-20]. Thirteen of the
twenty undergraduates who have worked on these projects and have since graduated are working on or have obtained a Master’s or Ph.D. in computer science or mathematics.

Dr. Bekmetjev has worked with several undergraduate research teams on pebbling problems from 2004-2011. These students presented their results at many national and state undergraduate research math conferences including MUMC (Michigan Undergraduate Math Conference), YMC (Young Mathematicians Conference), AMS JMM (American Mathematical Society Joint Mathematics Meeting), MathFest, SUMMR (Summer Undergraduate Michigan Mathematics Research), and others. Many of his research students went to graduate school in mathematics and computer science and became academic professionals in these fields.

The previous work in graph pebbling was successfully funded (in part) by HHMI grants that resulted in several publications in leading mathematical journals in this field [3, 12-13] and enhanced the curriculum in courses such as *Combinatorics and Graph Theory* and *Advanced Algorithms* offered by the departments of Mathematics and Computer Science at Hope College. In particular, Dr. Cusack and Dr. Bekmetjev developed a case study in graph pebbling that has been used in these courses.

1.7 Plans for External Funding To Continue Work

We plan to work with Dr. Darin Stephenson in the Mathematics department to submit an interdisciplinary NSF REU proposal focusing on graph theory and combinatorics, including our work in graph pebbling. We also plan to use some of the resources available from the Sponsored Research Office to identify other appropriate sources of funding.

1.8 Timeline

**Nov-April 2014** The PIs will perform a literature search. Each pertinent paper will be summarized and a potential further directions of study will documented and will be available online. The PIs will develop the detailed plan of research and prioritize the tasks.

**May-July 2014** Two students will join the project for 8 weeks. They will write a draft of their final results by the end of the 8 week period.

**Aug-Dec 2014** The students will present their research at departmental colloquia. The PIs will work with the students to finalize and submit a paper to an appropriate journal or conference.
2 Bibliography/References

poster presentation at Future Play 2007: The International Academic Conference on the
20. C. Martens, P. Mutreja, C.A. Cusack, “Using a Volunteer Computing Game to Solve the
Maximum Clique Problem,” poster presentation at Future Play 2006 International
Conference on the Future of Game Design and Technology, London, Ontario, Canada,
4 Biographical Sketches

Charles A. Cusack

Professional Preparation

<table>
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<tr>
<th>Institution</th>
<th>Major</th>
<th>Degree/Year</th>
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<tbody>
<tr>
<td>Michigan Technological University, Houghton, MI</td>
<td>Pure Mathematics</td>
<td>B.S. 1992</td>
</tr>
<tr>
<td>Michigan Technological University, Houghton, MI</td>
<td>Discrete Mathematics</td>
<td>M.S. 1994</td>
</tr>
<tr>
<td>University of Nebraska—Lincoln, Lincoln, NE</td>
<td>Computer Science</td>
<td>M.S. 1998</td>
</tr>
<tr>
<td>University of Nebraska—Lincoln, Lincoln, NE</td>
<td>Computer Science</td>
<td>Ph.D. 2000</td>
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Appointments

- Associate Professor, Computer Science, Hope College, 2013-present
- Assistant Professor, Computer Science, Hope College, 2007-2013
- Visiting Assistant Professor, Computer Science, Hope College, 2005-2007
- Software Engineer, Westshore Design, LLC, 2005-2006
- Lecturer, Computer Science and Engineering, University of Nebraska—Lincoln, 2003-2004
- J.D. Edwards Professor and Lecturer, Computer Science and Engineering, University of Nebraska—Lincoln, 2001-2003
- Lecturer, Computer Science and Engineering, University of Nebraska—Lincoln 2000-2001

Publications (student co-authors are indicated with *)

Most Closely Related


Other Significant

Synergistic Activities
1. Wrote An Active Introduction to Discrete Mathematics and Algorithms, a textbook for use in a new course I recently developed. This book is a combination of my own material with material from several open-source textbooks.
3. Program Committee member for The International Conference on Meaningful Play
4. Refined the Introduction to Computer Science Laboratory materials to increase student engagement and learning, added formal lab reports with a detailed grading rubric that increases the amount of useful feedback while reducing grading time.
5. Participated in REACH (Research Experience Across Cultures at Hope College) program by incorporating two high school students into my research group for six weeks during a summer.

Collaborators & Other Affiliations
Collaborators and Co-Editors
• Bekmetjev, Airat (Hope College)
• Edwards, Stephanie (Hope College)
• McFall, Ryan (Hope College)
• Parker, Darren (GVSU)
• Yurk, Brian (Hope College)

Graduate Advisors and Postdoctoral Sponsors
• Magliveras, Spyros (Florida Atlantic University, Ph.D. and M.S. advisor)
• Kreher, Donald (Michigan Technological University, M.S. advisor)
Airat Bekmetjev

PROFESSIONAL PREPARATION:

<table>
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<tr>
<th>Institution</th>
<th>Major</th>
<th>Degree &amp; Year</th>
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<tbody>
<tr>
<td>Moscow State University, Moscow,</td>
<td>Applied Mathematics</td>
<td>B.S. 1991</td>
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<tr>
<td>Russia</td>
<td></td>
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<tr>
<td>Arizona State University, Tempe,</td>
<td>Mathematics</td>
<td>Ph.D. 2002</td>
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<tr>
<td>AZ</td>
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APPOINTMENTS:
Associate Professor of Mathematics, Hope College, 2009 – present
Assistant Professor of Mathematics, Hope College, 2003 - 2009.
Assistant Professor, Gettysburg College, 2002 - 2003.
Teaching Assistant/Graduate Student, Arizona State University, 1996-2002.
Scientist/Researcher, Center for Computing and Applied Mathematics, Moscow, Russia, 1991-1995

SIGNIFICANT RELEVANT PUBLICATIONS:

OTHER PUBLICATIONS (student co-authors are indicated with *)

SYNERGISTIC ACTIVITIES:
1. Undergraduate research advisor. Every summer participate as an undergraduate research advisor in projects funded by NSF REU and NIH AREA15 programs
2. Graph Pebbling Interdisciplinary Case Study. Developed an interdisciplinary case study related to graph pebbling for use in the advanced Combinatorics and Graph Theory course.
3. REACH (Research Experience Across Cultures at Hope College). Participated in Hope College’s REACH program by being an advisor of a summer research team of high school students. The goal of the program is to create opportunities for students and teachers to engage in research projects with Hope's science and math faculty.
4. Conferences and workshops. Participated in organization of several student research conferences such as Spring Undergraduate Conference at Gettysburg College, Undergraduate Research Session at Pennsylvania MAA meeting, Undergraduate Research Conference at Moravian College. Participated as an invited speaker at Pew New Faculty Workshop at Hope College.

COLLABORATORS & OTHER AFFILIATIONS:

Collaborators:
Graham Brightwell (London School of Economics)
Andrzej Czygrinow (Arizona State University)
Glenn Hurlbert (Arizona State University)
Charles Cusack (Hope College)
Tim Pennings (Davenport University)
Todd Swanson (Hope College)
Brian Yurk (Hope College)
Nathan Tintle (Dordt College)
Jianhua Li (Hope College)
Kenneth Brown (Hope College)

Graduate and Post Doctoral Advisors:
Glenn Hurlbert (Arizona State University)
5 Current and Pending Support

Current:
None

Pending:
None